§1 Recap

X/k AV, $\mathcal{L}$ ample, $X^\vee := X/K_{\mathcal{L}}$

$$X \times X \longrightarrow X^\vee \times X$$

$$m^*\mathcal{L} \otimes p_1^*\mathcal{L}^{-1} \otimes p_2^*\mathcal{L}^{-1} \longmapsto P$$

$\circlearrowleft$ $K_{\mathcal{L}}$ — descent — Poincaré bundle

Thm $\forall\ \mathcal{M}$ on $S \times X$ $\in \mathrm{Pic}^{\circ}_{X/k}(S)$

s.th. $\mathcal{M}|_{S\times 0} \cong \mathcal{O}_S$

$\exists!\ S \xrightarrow{u} X^\vee$ s.th. $(u, \mathrm{id}_X)^* P = \mathcal{M}$.

Rmk Fact: S connected

Then $\mathcal{M} \in \mathrm{Pic}^{\circ}_{X/k}(S) \iff \mathcal{M}(s) \in \mathrm{Pic}^{\circ}(\kappa(s) \otimes_k X)$

for one $s \in S$

<u>Idea</u> On $(S \times X^\vee) \times X$ compare

$$p_{13}^* \mathcal{M} \quad \& \quad p_{23}^* P .$$

<u>Know</u> $\exists\ \Gamma \subseteq S \times X^\vee$ closed subscheme,

locus where two families agree

We showed $\Gamma \xrightarrow{\cong} S$, so $\Gamma = \Gamma_u$

for unique $u : S \to X^\vee$.

§2 Cohomology of Poincaré bundle

<u>Cor</u> $H^i(X \times X^\vee, P) = \begin{cases} 0 & i \neq g \\ k & i = g \end{cases}$

In pRic $\chi(P) = (-1)^g$.

<u>Proof</u> Seen last time

$$H^i(X^\vee \times X, P) = H^0(X^\vee, R^i p_{1,*} P)$$

supported at $\{0\} \subseteq X^\vee$.

If $0 \to K^0 \to \cdots \to K^g \to 0$ perfect complex of $\mathcal{O}_{X^\vee,0}$-modules computing $R^i p_{1,*} P$ universally,

$$0 \to \hat{K}^g \to \cdots \to \hat{K}^0 \to k \to 0$$

is an exact complex.

<u>Koszul complex</u> (R, m) reg loc ring of dim g.

$$m = (x_1, \ldots, x_g)$$

$$0 \to R \to R^{\binom{g}{1}} \to R^{\binom{g}{2}} \to \cdots \to R^{\binom{g}{g-1}} \to R \to R/m \to 0$$

R^g

$\Lambda^i(R^g)$

$R^{\binom{g}{k}}$ basis $e_{i_1} \wedge \cdots \wedge e_{i_k}$ w/ $1 \le i_1 < \cdots < i_k \le g$

Differentials

$$e_{i_1} \wedge \cdots \wedge e_{i_k} \mapsto -\sum_{j=1}^{k} (-1)^j x_{i_j} e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots$$

Fact: This is an exact complex, i.e. the complex resolves $k = R/m$.

E.g. $g=2$ $m = (x,y)$

$$0 \to R \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} R^2 \xrightarrow{(x,y)} R \xrightarrow{pr} R/(x,y) \to 0$$

Since any two proj resolutions of a module are homotopy equivalent,

$$\hat{K}^\bullet \sim \text{Koszul}(x_1, \dots, x_g)$$

$$(\text{case } R = \mathcal{O}_{X^\vee, 0})$$

Dual of Koszul complex is the Koszul complex, so

$$K^\bullet \sim \text{Koszul}(x_1, \dots, x_g)$$

$$\Rightarrow H^i(X^\vee \times X, P) = H^i(K^\bullet) = \begin{cases} 0 & i \neq g \\ k & i = g \end{cases}$$

∎

Cor $H^i(X, \mathcal{O}_X) \cong k^{\binom{g}{i}}$ $0 \le i \le g$

Proof Since $\mathcal{P}|_{\{0\} \times X} \cong \mathcal{O}_X$, so

$$H^i(X, \mathcal{O}_X) \cong H^i(K^\bullet(0)).$$

Koszul complex mod $\mathfrak{m} \subseteq \mathcal{O}_{X^\vee, 0}$ is just

$$0 \to k \xrightarrow{0} k^g \xrightarrow{0} \cdots \xrightarrow{0} k^g \xrightarrow{0} k \to 0$$

$\Rightarrow$ □

Rmk 1) Recall $\Omega^1_{X|k} \cong \mathcal{O}_X \otimes_k e^* \Omega^1_{X|k}$

$$\cong \mathcal{O}_X^g$$

$\Rightarrow$ Cor also computes all

$$H^i(X, \Omega^j_{X|k})$$

$$\Omega^j_{X|k} = \Lambda^j \, \Omega^1_{X|k}.$$

2) $k = \mathbb{C}$, then Hodge theory for $X = \mathbb{C}^g/\Lambda$

$$H^1(X, \mathbb{C}) \overset{\text{Hodge}}{\cong} H^1(X, \mathcal{O}_X) \oplus H^0(X, \Omega^1)$$

$$\begin{aligned} &\cong \\ &H^1((S^1)^{2g}, \mathbb{C}) \\ &\cong \\ &\mathbb{C}^{2g} \end{aligned} \Bigg\} \text{ from topology.}$$

$$+ \quad H^i(X, \mathbb{C}) = \Lambda^i_{\mathbb{C}} H^1(X, \mathbb{C})$$

Provide alternative computation of above cohomology groups.

§2 Duality & Degree

<u>Def</u> <u>Isogeny</u> $f : X \longrightarrow Y \underset{\text{def}}{=}$

surjective f w/ finite kernel

<u>Equivalent</u> f flat w/ finite kernel
(Miracle Flatness)

<u>or</u> f finite + locally free.

<u>Cor</u> $f : X \to Y$ isogeny, then

$f^\vee : Y^\vee \to X^\vee$ also isogeny and

$\deg f^\vee = \deg f$.

<u>Proof</u> $\mathcal{L}$ ample on Y. Then

$$t_x^* f^* \mathcal{L} \otimes f^* \mathcal{L}^{-1} = f^*\left(t_{f(x)}^* \mathcal{L} \otimes \mathcal{L}^{-1}\right)$$

i.e.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \phi_{f^*\mathcal{L}}}\downarrow & & \downarrow{\scriptstyle \phi_{\mathcal{L}}} \\ X^\vee & \xleftarrow{f^\vee} & Y^\vee \end{array}$$

commutes.

$\mathcal{L}$ ample, f finite, so $f^*\mathcal{L}$ ample.

$\Longrightarrow$ $\phi_{f^*\mathcal{L}}$ is isogeny

Thus $\ker f^\vee$ also finite, i.e. $f^\vee$ isogeny.

<u>Claim on degrees</u>

P_X on $X^\vee \times X$, P_Y on $Y^\vee \times Y$

By defn

$$(id_{Y^\vee}, f)^* P_Y \cong (f^\vee, id_X)^* P_X$$

Q_Y (=) on $Y^\vee \times X$ (=) Q_X

Want to argue:

$$\chi(Q_Y) \overset{?}{=} \deg(f) \cdot \chi(P_Y) = (-1)^g \cdot \deg f$$

$$\|$$

$$\chi(Q_X) \overset{?}{=} \deg(f^\vee) \cdot \chi(P_X) = (-1)^g \deg f^\vee$$

Prop G finite, X/k proper, $G \circlearrowright X$ freely

$\pi : X \to Y = X/G$ $\quad F$ coherent $\mathcal{O}_Y$-mod

Then $\quad \chi(\pi^* F) = |G| \cdot \chi(F)$

$$|G| = \deg \pi$$

Rmk Holds more generally for all finite étale $\pi : X \to Y$

(Reduce to above proposition.)

Compare $X \xrightarrow{\pi} Y$ map of curves.

Then genera of X, Y not just related by $\deg \pi$, but also ramification of π (Riemann-Hurwitz)

Proof Crucial input: $X \times_Y X \cong G \times X$

$$\Longrightarrow \quad \pi_* \pi^* \pi_* \mathcal{O}_X$$

$$= \pi_* \mathcal{O}_X \otimes_{\mathcal{O}_Y} \pi_* \mathcal{O}_X \cong \mathcal{O}_G \otimes_k \pi_* \mathcal{O}_X$$

$$\Longrightarrow \chi(\pi^* \pi_* \mathcal{O}_X) = \chi(\pi_* \pi^* \pi_* \mathcal{O}_X)$$

Seed case

$$= |G| \cdot \chi(\pi_* \mathcal{O}_X).$$

In fact, $\forall\ Z \subseteq Y$ closed

$$Z = (Z \times_Y X)/G$$

$\Longrightarrow$ Prop holds for all $\pi_* \mathcal{O}_{Z \times_Y X}$.

Use now Every $\mathcal{F}$ on Y can be obtained by successively forming kernels, cokernels and extensions from just the $\mathcal{O}_Z$, $Z \subseteq Y$ integral

+ χ additive on exact seq

+ π flat $\Rightarrow$ π^* is exact   cf. @ below

$\Longrightarrow$ Enough to prove   Prop for all

$\mathcal{O}_Z$, $Z \subseteq Y$.

By noetherian induction, may assume

Prop for F w/ dim supp $F \leq d$.

If dim $Z = d+1$, $I \subseteq \mathcal{O}_Z$ ideal,
(Z integral),

get $0 \to I \to \mathcal{O}_Z \to \mathcal{O}_{V(I)} \to 0$

So prop for I equivalent to prop for $\mathcal{O}_Z$.

So <u>to show</u>:   Assume prop holds if
dim Supp $F \leq d$

Then $\forall$ Z of dim $d+1$, Z integral, There is a sheaf of ideals $0 \neq \mathcal{I} \subseteq \mathcal{O}_Z$ for which the prop holds.

Now use seed case: $\eta \in Z$ gen pt

$$\underbrace{(\pi_* \mathcal{O}_{Z \times X})}_{\mathcal{F}}{}_\eta \text{ is a } \mathcal{O}_{Z,\eta}\text{-vector space.}$$

Pick basis $s_1, \ldots, s_r$ $\quad r = |G|$

Pick open U s.th. s_i extend to

$$s_i \in \mathcal{F}(U).$$

$$\rightsquigarrow \quad \mathcal{O}_U^r \xrightarrow{\varphi} \mathcal{F}|_U$$

Exercise $\mathcal{I} \subseteq \mathcal{O}_Z$ ideal sheaf s.th.

$$V(\mathcal{I}) = Z \setminus U.$$

Then $\forall N \gg 0$, φ extends (uniquely) to $\tilde{\varphi} : (\mathcal{I}^N)^r \longrightarrow \mathcal{F}$.

In exact seq

$$0 \longrightarrow (\mathcal{I}^N)^r \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\operatorname{Im}\tilde{\varphi} \longrightarrow 0,$$

prop holds for $\mathcal{F}$ & $\mathcal{F}/\operatorname{Im}\tilde{\varphi}$,

hence holds for $\mathcal{I}^N$. ∎

Prop & Cor

<u>Repeatedly used</u> Given ex seq of coh $\mathcal{O}_X$-modules

@

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

If Prop holds for two, it does for third.

(Uses π^* exact since π flat)

<u>Used at beginning</u> $A \longrightarrow B$

Then $B \otimes_A B = B \otimes_A B$

(viewed as B-module) $\overset{\pi_*}{\longmapsto}$ (view as A-module)

§3 Symmetry $(X^\vee)^\vee = X$

More precisely, P on $X^\vee \times X$ also has universal property for $X^\vee$.

Prop $X, Y / k$ AVs, same dimension

Q lb on $X \times Y$ s.th.

$$Q|_{0 \times Y} \cong \mathcal{O}_Y, \quad Q|_{X \times 0} \cong \mathcal{O}_X.$$

Equivalent

1) Q only trivial over $\{0\} \subseteq X$ via p_1

2) ——————— $\{0\} \subseteq Y$ via p_2

3) $|\chi(Q)| = 1$

In this case, $X \underset{Q}{\cong} Y^\vee$, $Y \underset{Q}{\cong} X^\vee$.

<u>Proof</u> Enough to show 2) $\Leftrightarrow$ 3).

Assume 2): Fiber $X \times \{0\} \in \mathrm{Pic}^\circ(X)$

$$\Rightarrow \quad Q \in \mathrm{Pic}^\circ_{X|k}(Y).$$

$$\Rightarrow \exists!\ u : Y \to X^\vee \text{ s.th.}$$

$$Q = (u, id_X)^* P.$$

By 2), $\ker(u) = \{0\}$.

Since $\dim Y = \dim X = \dim Y^\vee$, u is iso.

$$\Rightarrow |\chi(Q)| = |\chi(P)| = 1.$$

<u>Conversely</u> Assume $|\chi(Q)| = 1$. If

u finite, then prev. Prop applies

and $\quad |\chi(Q)| = \deg u \cdot |\chi(P)|$

$\Rightarrow$ $\deg u = 1$ i.e. is iso.

$\Rightarrow$ $\ker(u) = \{0\}$ which is 2).

If $\dim \ker u > 0$, pick any finite

$K \subseteq \ker u$.

Then u factors over $Y \to Y/K$.

$\Rightarrow$ $\deg K \mid |X(\mathbb{Q})|$ ↯
by Prop again

$\Rightarrow$ $\square$

<u>Cor</u> $\forall X$, the canonical

$X \to X^{\vee\vee}$ is an isomorphism.

Given $f : X \to Y$, $(f^\vee)^\vee = f$.

<u>Proof</u> For $X \xrightarrow{\cong} (X^\vee)^\vee$, this is the prev. Prop.

For $f = (f^\vee)^\vee$, use that

$$(\mathrm{id}_{Y^\vee}, f)^* P_Y = (f^\vee, \mathrm{id}_X)^* P_X$$

on $X \times Y^\vee$. $\square$

§4 Poincaré Reducibility

<u>Prop</u> (Poincaré) Given $Y \subseteq X$ AVs, $\exists\ Z \subseteq X$ s.th.

$$Y \times Z \longrightarrow X$$
$$(y, z) \longmapsto y + z$$

is an isogeny.

<u>Proof</u> $i : Y \hookrightarrow X$, $i^\vee : X^\vee \longrightarrow Y^\vee$

$\mathcal{L}$ ample on X, $Z := \phi_{\mathcal{L}}^{-1}(\ker i^\vee)$

$$Y \cap Z = \{ y \in Y \mid (t_y^* \mathcal{L} \otimes \mathcal{L}^{-1})|_Y \cong \mathcal{O}_Y \}$$

$$= K(\mathcal{L}|Y).$$

$\mathcal{L}$ ample $\Longrightarrow$ $K_{\mathcal{L}|Y}$ finite

$\Longrightarrow$ $\ker(Y \times Z \longrightarrow X)$ finite.

<u>Moreover</u> $i^\vee$ is surjective.

Namely if $i^\vee(X^\vee) \subsetneq Y^\vee$, then

$$\begin{array}{ccc} i = (i^\vee)^\vee : Y & \longrightarrow & \left[i^\vee(X^\vee)\right]^\vee \\ & \searrow^{i} & \downarrow \\ & & X \end{array}$$

Since $\dim \left(i^\vee(X^\vee)\right)^\vee = \dim i^\vee(X^\vee)$,

i can only be injective if

$i^\vee(X^\vee) = Y^\vee$. In part., $\dim Z$

$= \dim i^\vee(X^\vee) = \dim X - \dim Y.$

Left to show

Claim 1 Z°_{red} := connected comp of 0

w/ reduced scheme str is an AV.

To show it is geometrically reduced,

the pt is smooth group scheme.

Proof $Z^\circ \subseteq Z$ conn. comp. of 0.

This is a group scheme.

$p = \operatorname{char} k$

If $p \nmid n$, then $Z^\circ[n] \subseteq X[n]$ is a

subgroup of étale group scheme,

hence étale itself.

$\Longrightarrow$ $Z^\circ[n] \subseteq Z^\circ_{red}$.

<u>Claim 2</u> $\bigcup_{p \nmid n} Z^\circ[n]$ is <u>dense</u> in Z°_{red}.

i.e. $Z^\circ_{red} = \bigcap_{\substack{T \subseteq Z^\circ_{red} \text{ closed},\ Z[n] \subseteq T \\ \forall p \nmid n.}} T$ $= V(I)$ @

I = fcts f on Z°_{red} s.th. $f|_{Z^\circ[n]} = 0$ $\forall n$.

Assume claim. Then $\forall\, U \subseteq Z^\circ_{red}$ open,

$$\mathcal{O}_{Z^\circ_{red}}(U) \hookrightarrow \prod_{p \nmid n} \mathcal{O}_{Z^\circ[n]}(U \cap Z^\circ[n])$$

(Is reformulation of $I = 0$ in @.)

$$\Longrightarrow \bar{k} \otimes_k \mathcal{O}_{Z^\circ_{red}}(U) \hookrightarrow \prod_{p \nmid n} \bar{k} \otimes \ldots$$

But RHS stays reduced since $Z^\circ[n]$ are étale.

$\Longrightarrow \mathcal{O}_{Z^\circ_{red}}(U)$ geometrically reduced $\forall U$

$\Rightarrow$ ☐ Claim 1

Proof of Claim 2

$Z^\circ[n]_{\bar{k}} \subseteq (Z^\circ_{\bar{k}})_{red}$ since $Z^\circ[n]$ étale.

$\bigcup_{p \nmid n} Z^\circ[n]_{\bar{k}}$ stable under addition & inverses, so its closure W in $(Z^\circ_{\bar{k}})_{red}$ is a group scheme.

Since $\bar{k}$ alg closed, W°_{red} is then an AV. But $(Z^\circ_{\bar{k}})_{red}$ is also AV and

$$|W^\circ_{red}[n]| = |Z^\circ[n]| = |(Z^\circ_{\bar{k}})_{red}[n]|.$$

So $W^\circ_{red} = (Z^\circ_{\bar{k}})_{red}$

since $(\text{AV of dim } g)[n] \cong (\mathbb{Z}/n)^{2g}$.

Since $\mathcal{O}_{Z^\circ_{red}} \hookrightarrow \mathcal{O}(Z^\circ_K)_{red}$

$\Longrightarrow \bigcup_{p \nmid n} Z^\circ[n]$ dense in Z°_{red}.

$\square$

Claim 2.

<u>End of proof</u>

$Y \times Z^\circ_{red} \longrightarrow X$ is an isogeny as in the prop. $\square$